

# Variationally equivalent problems and variations of Noether currents

M. Francaviglia, M. Palese and E. Winterroth

Department of Mathematics, University of Torino

via C. Alberto 10, I-10123 Torino, Italy

E-MAIL: [MARCELLA.PALESE, EKKEHART.WINTERROTH]@UNITO.IT

## Abstract

We consider systems of local variational problems defining non vanishing cohomology classes. In particular, we prove that the conserved current associated with a generalized symmetry, assumed to be also a symmetry of the variation of the corresponding local inverse problem, is variationally equivalent to the variation of the strong Noether current for the corresponding local system of Lagrangians. This current is conserved and a sufficient condition will be identified in order such a current be global.

**2000 MSC:** 55N30, 55R10, 58A12, 58A20, 58E30, 70S10.

**Key words:** jets; gauge-natural bundles; local variational problem; variation of currents.

## 1 Introduction

The aim of this paper is the study of Noether conservation laws associated with some ‘variational’ invariance of global Euler-Lagrange morphisms of local problems of a given type. In this context, the question arises whether we would be interested in conservation laws different from those directly associated with invariance properties of field equations. The answer to this question relays on Emmy Noether’s famous and celebrated paper *Invariante Variationsprobleme* [16]. As it is well known, Noether’s paper was motivated by the fact that, although the gravitational field equations were global, the associated conservation laws found by Einstein by a nonvariational approach were not (think of the well known energy-momentum *pseudo-tensor*). The underlying idea was that of looking for conservation laws coming from invariance properties of a (*possibly local*) Lagrangian (rather than a field equation solely) to find a way of associating global conservation laws with the gravitational *field*.

Explicitly, in the introduction of her paper, Noether wrote: *Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden; so*

underlining the relevance of the study of differential equations generated by an invariant variational problem in its whole rather than of arbitrary differential equations admitting a Lie group of symmetries. It is pointed out that considering invariance of variational problems the issue is a major refinement in the results: to symmetries of equations could correspond conservation laws which have a nonvariational meaning and thus could not be characterized in a similar precise manner.

In fact, contrarily to what sometime improperly stated, Noether's Theorem (I and II Theorems actually) were not 'formulated for Euler-Lagrange equations in field theory', rather they are statements about the invariance of the variational problem (in other words, of the action integral) with respect to a finite continuous group of transformations and an infinite continuous group of transformations, respectively. The direct object of Noether's investigations are what she calls *Lagrangeschen Ausdrücke; d.h. die linken Seiten der Lagrangeschen Gleichungen*, which we shall call hereafter Euler-Lagrange expressions. The accent is not put on field *equations* although her results have, of course, *also* consequences concerning invariance properties of equations. It is maybe noteworthy that all Noether considerations are made 'off shell', *i.e.* not along solutions of Euler-Lagrange equations; in particular it is an evident matter that, 'on shell' Noether identities would reduce to the trivial identity  $0 = 0$ . It is also important to stress that Noether immediately consider the formulation of a variational problem at an infinitesimal level (*integralfreie Identität*).

Noether's Theorem II is in fact concerned with a variation of the Euler-Lagrange expressions. Symmetry properties of the Euler-Lagrange expressions play a fundamental role since they introduce *a cohomology class which adds up to Noether currents*<sup>1</sup>; they are related with invariance properties of the first variation, thus with the vanishing of a second variational derivative. As we shall see later on, the concept of a variation of Noether current is then clearly involved. In this paper, therefore, symmetries of Euler-Lagrange morphisms or, more generally, of so-called variational 'dynamical forms' are considered inasmuch as they can provide informations about Noether currents of some potential Lagrangians, also in the spirit of the Bessel-Hagen version of Noether's Theorem II [3]. We would like to stress that a Noether current is not necessarily a *conserved current*. We shall investigate under which conditions the variation of a strong Noether current is conserved: the vanishing of the second variational derivative will be involved; but an additional sufficient condition will be identified in order such a current be global.

Geometric definitions of conserved quantities in field theories have been proposed within formulations based on symmetries of Euler-Lagrange operator rather than of the Lagrangian (see *e.g.* [22, 2]). In particular, a definition of *variation of conserved currents* for gauge-natural theories [6] has been proposed in [10] where the first variation of a 'deformed' (also called there 'variational') Lagrangian – obtained by contracting the Euler-Lagrange term with the infinitesimal symmetry defined by preferred lifts of infinitesimal principal

---

<sup>1</sup>A formulation in modern language of Noether's results.

automorphisms (gauge-natural lifts) – was considered. On the other hand, without the fixing of a connection *a priori*, the gauge-natural second variational derivative of gauge-natural invariant Lagrangian has been exploited in order to construct Noether covariant conserved current [17, 18], by showing the relationship of the Bergmann–Bianchi morphism with the variational derivative of a conveniently deformed (gauge-natural) Lagrangian and with Noether identities: generalized symmetries and some version of Noether’s Theorem II play a fundamental role [8, 9]; see also [23] for a characterization of superpotentials in gauge theories. Strictly related with the present approach are also papers proposing the concept of relative conservation laws; see *e.g.* [7].

Inspired by Noether’s results and by Lepage’s seminal paper [15], geometrical formulations of the calculus of variations on fibered manifolds constitute a large class of theories for which the Euler–Lagrange operator is a morphism of an exact sequence, see *e.g.* [1, 5, 14, 24, 25, 26]. The module in degree  $(n + 1)$ , contains so-called (variational) dynamical forms; a given equation is globally an Euler–Lagrange equation if its dynamical form is the differential of a Lagrangian and this is equivalent to the dynamical form being closed in the complex, *i.e.* Helmholtz conditions hold true, and its cohomology class being trivial. Dynamical forms which are only *locally variational*, *i.e.* which are closed in the complex and define a non trivial cohomology class, admit a system of local Lagrangians, one for each open set in a suitable covering, which satisfy certain relations among them. Global projectable vector field on a jet fiber manifold which are symmetries of dynamical forms, in particular of locally variational dynamical forms, and corresponding formulations of Noether theorem II can be considered in order to determine obstructions to the globality of associated conserved quantities [11]. Analogously, the concept of global (and local) variationally trivial Lagrangians and, in general, of variationally trivial currents (*i.e.*  $(n - 1)$ -forms) will be fundamental in this paper.

In fact, in the sequel, a *locally variational form* is any closed  $p$ -form in the variational sequence; we will be faced with inverse problems at any degree of variational forms.

The role, in the cohomology, of the *variational Lie derivative*, a differential operator acting on equivalence classes of forms in the variational sequence, has very relevant implications in the calculus of variations. Iterated variational derivatives define higher order variations [12, 17, 19]; thus variations of currents can be recognized in this approach. Consequently, we shall now study variations of conserved currents in a quite general setting.

## 2 Equivalence of local variational problems

We shall consider the variational sequence [14] defined on a fibered manifold  $\pi : \mathbf{Y} \rightarrow \mathbf{X}$ , with  $\dim \mathbf{X} = n$  and  $\dim \mathbf{Y} = n + m$ . For  $r \geq 0$  we have the  $r$ -jet space  $J_r \mathbf{Y}$  of jet prolongations of sections of the fibered manifold  $\pi$ . We have also the natural fiberings  $\pi_s^r : J_r \mathbf{Y} \rightarrow J_s \mathbf{Y}$ ,  $r \geq s$ , and  $\pi^r : J_r \mathbf{Y} \rightarrow \mathbf{X}$ ; among these the fiberings  $\pi_{r-1}^r$  are *affine bundles* which induce the natural fibered

splitting

$$J_r \mathbf{Y} \times_{J_{r-1} \mathbf{Y}} T^* J_{r-1} \mathbf{Y} \simeq J_r \mathbf{Y} \times_{J_{r-1} \mathbf{Y}} (T^* \mathbf{X} \oplus V^* J_{r-1} \mathbf{Y}),$$

which, in turn, induces also a decomposition of the exterior differential on  $\mathbf{Y}$  in the *horizontal* and *vertical differential*,  $(\pi_r^{r+1})^* \circ d = d_H + d_V$ . By  $(j_r \Xi, \xi)$  we denote the jet prolongation of a *projectable vector field*  $(\Xi, \xi)$  on  $\mathbf{Y}$ , and by  $j_r \Xi_H$  and  $j_r \Xi_V$  the horizontal and the vertical part of  $j_r \Xi$ , respectively.

We have the *sheaf splitting*  $\mathcal{H}_{(s+1,s)}^p = \bigoplus_{t=0}^p \mathcal{C}_{(s+1,s)}^{p-t} \wedge \mathcal{H}_{s+1}^t$  where  $\mathcal{H}_{(s,q)}^p$  and  $\mathcal{H}_s^p$  ( $q \leq s$ ) are sheaves of *horizontal forms*, and  $\mathcal{C}_{(s,q)}^p \subset \mathcal{H}_{(s,q)}^p$  are subsheaves of *contact forms* [14]. Let us denote by  $h$  the projection onto the nontrivial summand with the highest value of  $t$  and by  $d \ker h$  the sheaf generated by the corresponding presheaf and set then  $\Theta_r^* \equiv \ker h + d \ker h$ ; the quotient sequence

$$0 \rightarrow \mathcal{R}_{\mathbf{Y}} \rightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} \Lambda_r^n / \Theta_r^n \xrightarrow{\mathcal{E}_n} \Lambda_r^{n+1} / \Theta_r^{n+1} \xrightarrow{\mathcal{E}_{n+1}} \Lambda_r^{n+2} / \Theta_r^{n+2} \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

defines the *r-th order variational sequence* associated with the fibered manifold  $\mathbf{Y} \rightarrow \mathbf{X}$ ; here  $\Lambda_s^p$  is obviously the standard sheaf of  $p$ -forms on  $J_s \mathbf{Y}$ . The quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves  $\mathcal{V}_r^k$  of  $k$ -forms on jet spaces of higher order. In particular, currents are classes  $\nu \in (\mathcal{V}_r^{n-1})_{\mathbf{Y}}$ ; Lagrangians are classes  $\lambda \in (\mathcal{V}_r^n)_{\mathbf{Y}}$ , while  $\mathcal{E}_n(\lambda)$  is called a Euler–Lagrange form (being  $\mathcal{E}_n$  the Euler–Lagrange morphism); dynamical forms are classes  $\eta \in (\mathcal{V}_r^{n+1})_{\mathbf{Y}}$  and  $\mathcal{E}_{n+1}(\eta)$  is a Helmholtz form (being  $\mathcal{E}_{n+1}$  the corresponding Helmholtz morphism).

It turns out that the variational sequence is an exact resolution of the constant sheaf  $\mathcal{R}_{\mathbf{Y}}$  over  $\mathbf{Y}$ . Since the variational sequence is a soft resolution of the constant sheaf  $\mathcal{R}_{\mathbf{Y}}$  over  $\mathbf{Y}$ , the cohomology of the complex of global sections, denoted by  $H_{VS}^*(\mathbf{Y})$ , is naturally isomorphic to both the Čech cohomology of  $\mathbf{Y}$  with coefficients in the constant sheaf  $\mathcal{R}$  and the de Rham cohomology  $H_{dR}^* \mathbf{Y}$  [14].

Let  $\mathbf{K}_r^p \doteq \text{Ker } \mathcal{E}_p$ . We have the short exact sequence of sheaves

$$0 \rightarrow \mathbf{K}_r^p \xrightarrow{i} \mathcal{V}_r^p \xrightarrow{\mathcal{E}_p} \mathcal{E}_p(\mathcal{V}_r^p) \rightarrow 0.$$

For any global section  $\beta \in (\mathcal{V}_r^{p+1})_{\mathbf{Y}}$  we have  $\beta \in (\mathcal{E}_p(\mathcal{V}_r^p))_{\mathbf{Y}}$  if and only if  $\mathcal{E}_{p+1}(\beta) = 0$ , which are conditions of local variability. A global inverse problem is to find necessary and sufficient conditions for such a locally variational  $\beta$  to be globally variational. In particular,  $\mathcal{E}_n(\mathcal{V}_r^n)$  is the sheaf of Euler–Lagrange morphisms and  $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_{\mathbf{Y}}$  if and only if  $\mathcal{E}_{n+1}(\eta) = 0$ , which are Helmholtz conditions.

The above exact sequence gives rise to the long exact sequence in Čech cohomology

$$0 \rightarrow (\mathbf{K}_r^p)_{\mathbf{Y}} \rightarrow (\mathcal{V}_r^p)_{\mathbf{Y}} \rightarrow (\mathcal{E}_p(\mathcal{V}_r^p))_{\mathbf{Y}} \xrightarrow{\delta_p} H^1(\mathbf{Y}, \mathbf{K}_r^p) \rightarrow 0.$$

In particular, every  $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_{\mathbf{Y}}$  (*i.e.* locally variational) defines a cohomology class  $\delta(\eta) \equiv \delta_n(\eta) \in H^1(\mathbf{Y}, \mathbf{K}_r^n) \simeq H_{VS}^{n+1}(\mathbf{Y}) \simeq H^{n+1}(\mathbf{Y}, \mathcal{R})$ . Furthermore, every  $\mu \in (d_H(\mathcal{V}_r^{n-1}))_{\mathbf{Y}}$  (*i.e.* locally variationally trivial) defines a cohomology class  $\delta'(\mu) \equiv \delta_{n-1}(\mu) \in H^1(\mathbf{Y}, \mathbf{K}_r^{n-1}) \simeq H_{VS}^n(\mathbf{Y}) \simeq H^n(\mathbf{Y}, \mathcal{R})$ .

For any countable open covering  $\mathfrak{U} = \{U_i\}_{i \in I}$ ,  $I \subset \mathbb{Z}$ ,  $C^q(\mathfrak{U}, \mathcal{V}_r^p)$  is the set of  $q$ -cochains with coefficients in the sheaf  $\mathcal{V}_r^p$ , if  $\mathfrak{d} : C^q(\mathfrak{U}, \mathcal{V}_r^p) \rightarrow C^{q+1}(\mathfrak{U}, \mathcal{V}_r^p)$  is the *coboundary operator*, the *connecting homomorphism*  $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}_p^{-1}$  is the mapping of cohomologies in the corresponding diagram of cochain complexes.

Note that  $\eta$  is globally variational if and only if  $\delta(\eta) = 0$ . In the following we will be interested in the non trivial case  $\delta(\eta) \neq 0$  whereby  $\eta = \mathcal{E}_n(\lambda)$  can be solved only locally, *i.e.* for any countable good covering of  $\mathbf{Y}$  there exists a local Lagrangian  $\lambda_i$  over each subset  $U_i \subset \mathbf{Y}$  such that  $\eta_i = \mathcal{E}_n(\lambda_i)$ .

A *local variational problem* is a system of local sections  $\lambda_i$  of  $(\mathcal{V}_r^n)_{U_i}$  such that  $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$ . We notice that every cohomology class in  $H^{n+1}(\mathbf{Y}, \mathbb{R})$  gives rise to local variational problems. We are then faced with the situation that  $\lambda = \{\lambda_i\}_{i \in I}$  is a 0-cochain of Lagrangians in Čech cohomology with values in the sheaf  $\mathcal{V}_r^n$ , *i.e.*  $\lambda \in C^0(\mathfrak{U}, \mathcal{V}_r^n)$ ; we shall denote by  $\eta_\lambda$  the 0-cochain formed by the restrictions  $\eta_i = \mathcal{E}_n(\lambda_i)$  (and so will do at any degree of forms). Of course, if  $\eta \in C^0(\mathfrak{U}, \mathcal{V}_r^{n+1})$ , then  $\mathfrak{d}\eta = 0$  if and only if  $\eta$  is global. Note that  $\mathfrak{d}\lambda = 0$  implies  $\mathfrak{d}\eta_\lambda = 0$ , while we only have  $\mathfrak{d}\eta_\lambda = \eta_{\mathfrak{d}\lambda} = 0$  *i.e.*  $\mathfrak{d}\lambda \in C^1(\mathfrak{U}, \mathbf{K}_r^n)$  [4].

**Remark 1** Two local variational problems of degree  $p$  are *equivalent* if and only if they give rise to the same variational class of forms as the image of the corresponding morphism  $\mathcal{E}_p$  in the variational sequence.  $\square$

Noether's Theorems relate symmetries of a variational problem to conserved quantities: the concept of a *variational Lie derivative* operator  $\mathcal{L}_{j_r \Xi}$  which is a local differential operator enables us to define symmetries of Lagrangian and dynamical forms (as well as of higher degree classes of forms in the variational sequence) and corresponding Noether's Theorems [13]. We notice that the variational Lie derivative sends a diagram of cochain complexes into a diagram of cochain complexes and thus acts on cohomology classes. The cohomology class defined by a system of local Lagrangian is related with the cohomology class defined by the local variational problem given by the system of their local variational Lie derivative: closed variational forms defining nontrivial cohomology classes are transformed in variational forms with trivial cohomology classes [20, 21].

We call  $(\{U_i\}_{i \in \mathbb{Z}}, \lambda_i)$  a *presentation* of the local variational problem and we remark that an infinitesimal symmetry of a local presentation is not necessarily a symmetry of another local presentation [11].

Let us now first summarize some preliminary results, partially stated in [20, 21].

**Lemma 1** Let  $\mu \in \mathcal{V}_r^p$ , with  $p \leq n$ , be a *locally variationally trivial*  $p$ -form, *i.e.* such that  $\mathcal{E}_p(\mu) = 0$  and let  $\delta_p(\mu_\nu) \neq 0$ . We have  $\delta_p(\mathcal{L}_\Xi \mu_\nu) = 0$ . Analogously, let  $\eta \in \mathcal{V}_r^p$ , with  $p \geq n+1$ , be a *locally variational*  $p$ -form, *i.e.* such that  $\mathcal{E}_p(\eta) = 0$  and let  $\delta_p(\eta_\lambda) \neq 0$ . We have  $\delta_p(\mathcal{L}_\Xi \eta_\lambda) = 0$ .

PROOF. In fact, we have a 0-cocycle of currents  $\nu_i$  ( $\mathfrak{d}\nu_i \neq 0$ ) such that  $\mu = d_H \nu_i$  and  $\mathfrak{d}\mu_\nu = 0$ ; by using the representation of the Lie derivative of

classes of variational forms of degree  $p \leq n - 1$  given in [13], we have

$$\mu_{\mathcal{L}_{\Xi}\nu_i} = d_H(\Xi_H \rfloor d_H\nu_i + \Xi_V \rfloor d_V\nu_i)$$

on the other hand, resorting to the naturality of the variational Lie derivative, we have  $\mu_{\mathcal{L}_{\Xi}\nu_i} = \mathcal{L}_{\Xi}\mu_{\nu}$ , so that  $\delta_p(\mathcal{L}_{\Xi}\mu_{\nu}) = 0$ .

Analogously, we have a 0-cocycle of Lagrangians (case  $p = n + 1$ ) – or of variational forms of higher degree (in case  $p = n + 2$  we have a 0-cocycle of dynamical forms) –  $\lambda_i$  ( $\mathfrak{d}\lambda_i \neq 0$ ) such that  $\eta = \mathcal{E}_p(\lambda_i)$ ; by linearity

$$\eta_{\mathcal{L}_{\Xi}\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_{\lambda}),$$

and again resorting to the naturality of the variational Lie derivative we have  $\eta_{\mathcal{L}_{\Xi}\lambda_i} = \mathcal{L}_{\Xi}\eta_{\lambda}$ , so that  $\delta_p(\mathcal{L}_{\Xi}\eta_{\lambda}) = 0$ .

**Remark 2** In particular, since we also have  $\mathcal{L}_{\Xi}\mu_{\nu} = \mu_{\mathcal{L}_{\Xi}\nu_i} = d_H(\Xi_H \rfloor \mu_{\nu} + \Xi_V \rfloor p_{d_V\mu_{\nu}})$ , from the definition of an equivalent variational problem, we can state that the local problem defined by  $\mathcal{L}_{\Xi}\nu_i$  is variationally equivalent to the global problem defined by  $\Xi_H \rfloor \mu_{\nu} + \Xi_V \rfloor p_{d_V\mu_{\nu}}$ . The class  $\mu_{\nu} = d_H\nu_i$  is assumed to satisfy  $\mathfrak{d}\mu_{\nu} = 0$ , *i.e.* it is a global object. We also notice that  $\Xi_V \rfloor d_V\nu_i = \Xi_V \rfloor p_{d_V\mu_{\nu}} + d_H\phi_i$  thus  $\Xi_V \rfloor d_V\nu_i - d_H\phi_i$  is a global object and, in general, its cohomology class can be non trivial.

Analogously, as a consequence of the fact that  $\eta_{\mathcal{L}_{\Xi}\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_{\lambda})$  we have that the local problem defined by the local presentation  $\mathcal{L}_{\Xi}\lambda_i$  is variationally equivalent to the global problem defined by  $\Xi_V \rfloor \eta_{\lambda}$ .  $\square$

## 2.1 Variation vector fields which are generalized symmetries

Let  $\eta_{\lambda}$  be the global Euler–Lagrange morphism of a local variational problem. In the sequel, we will assume  $\Xi$  to be a generalized symmetry, *i.e.* a symmetry of a class of  $(n + 1)$ -forms  $\eta$  in the variational sequence.

It is a well known fact that  $\Xi$  being a generalized symmetry implies that  $\mathcal{E}_n(\Xi_V \rfloor \eta) = 0$ , thus locally  $\Xi_V \rfloor \eta = d_H\nu_i$ , then there exists a 0-cocycle  $\nu_i$ , defined by  $\mu_{\nu} = \Xi_V \rfloor \eta_{\lambda} \equiv d_H\nu_i$ . Notice that  $\mathfrak{d}(\Xi_V \rfloor \eta_{\lambda}) = 0$ , but in general  $\delta_n(\Xi_V \rfloor \eta_{\lambda}) \neq 0$  [11]. Along critical sections this implies the conservation law  $d_H\nu_i = 0$ .

On the other hand Noether’s Theorem II implies that locally  $\mathcal{L}_{\Xi}\lambda_i = d_H\zeta_i$ , thus we can write  $\Xi_V \rfloor \eta_{\lambda} + d_H(\epsilon_i - \zeta_i) = 0$ , where  $\epsilon_i = j_r \Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i$  is the usual *canonical* Noether current; the current  $\epsilon_i - \zeta_i$  is a *local* object and it is conserved along the solutions of Euler–Lagrange equations (critical sections). Note that the Noether current  $\epsilon_i$  is conserved along critical sections if and only if  $\Xi$  is also a symmetry of  $\lambda_i$ . It is clear that, without additional information, to determine the current  $\nu_i$  is not a simple matter, specifically when  $\Xi$  is *not* also a symmetry of the Lagrangian. We also stress that when  $\Xi$  is only a symmetry of a dynamical form and not a symmetry of the Lagrangian, the current  $\nu_i + \epsilon_i$

is not a conserved current and it is such that  $d_H(\nu_i + \epsilon_i)$  is locally equal to  $d_H\zeta_i$ ; see also [22]. We shall call  $(\nu_i + \epsilon_i)$  a *strong Noether current*. Notice that if  $\Xi$  would be also a symmetry of the cochain of Lagrangians a strong Noether current would turn out to be a *conserved current* along any sections, not only along critical sections.

We can associate a system of global currents with a system of local conserved currents by taking the Lie derivatives of the local system, for which  $\delta'(\mathcal{L}_\Xi \mu_\nu) = \delta'(\mathcal{L}_\Xi(\Xi_V \rfloor \eta_\lambda)) = \delta'(\mathcal{L}_\Xi(d_H \nu_i)) = \delta'(d_H(\mathcal{L}_\Xi \nu_i)) = 0$  holds true. As a result we have that divergence expressions of the local problem defined by  $\mathcal{L}_\Xi \nu_i$  coincide with divergence expressions for the global current  $\Xi_H \rfloor \Xi_V \rfloor \eta_\lambda + \Xi_V \rfloor p_{d_V(\Xi_V \rfloor \eta_\lambda)}$ .

We shall now study variations of conserved currents in a quite general setting by determining the condition for the variation of a system of local strong Noether current to be equivalent to a system of *global conserved currents*.

**Lemma 2** *Suppose that  $\mathfrak{d}\lambda_i = d_H\gamma_{ij}$ . The condition  $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$  implies  $\mathcal{L}_\Xi d_H\gamma_{ij} = \mathfrak{d}d_H\beta_i$ , with  $\beta_i = \nu_i + \epsilon_i + d_H\omega_i$ .*

PROOF. Since, by linearity, we have  $\mathcal{L}_\Xi \mathfrak{d}\mathcal{L}_\Xi \lambda_i = \mathfrak{d}\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i$ , the condition  $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$  implies  $\mathcal{L}_\Xi \mathfrak{d}\mathcal{L}_\Xi \lambda_i = 0$ , i.e. by Noether's Theorem II we must have  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = d_H\zeta_{ij}$ , where  $\zeta_{ij}$  is the sum of the Noether current associated with  $\mathfrak{d}\lambda_i$  and a form locally given as  $\mathfrak{d}\nu_i + d_H\rho_{ij}$ . On the other hand  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = \Xi_V \rfloor \mathcal{E}_n(\mathfrak{d}\lambda_i) + d_H(j_{r+1}\Xi_V \rfloor p_{d_V \mathfrak{d}\lambda_i} + \xi \rfloor \mathfrak{d}\lambda_i) \equiv \mathfrak{d}d_H\epsilon_i$ , where  $\epsilon_i$  is the Noether current associated with  $\lambda_i$ .

Since we are assuming  $\mathfrak{d}\lambda_i = d_H\gamma_{ij}$ , we also have  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = \mathcal{L}_\Xi d_H\gamma_{ij}$ . On the other hand, since we assume  $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$  we also have by Noether's Theorem II  $\mathcal{L}_\Xi \lambda_i = d_H\beta_i$ , hence from  $\mathfrak{d}\mathcal{L}_\Xi \lambda_i = \mathcal{L}_\Xi \mathfrak{d}\lambda_i$  we get, locally,  $\mathfrak{d}d_H\beta_i = \mathcal{L}_\Xi d_H\gamma_{ij}$ , i.e. (by Noether's Theorem II we actually have  $\beta_i = \nu_i + \epsilon_i + d_H\omega_i$ )  $\mathfrak{d}d_H(\nu_i + \epsilon_i) = \mathcal{L}_\Xi d_H\gamma_{ij}$ . Thus  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = d_H\zeta_{ij} = \mathcal{L}_\Xi d_H\gamma_{ij} = \mathfrak{d}d_H\beta_i$ .

**Remark 3** The above is also equivalent to the equality  $\mathfrak{d}\epsilon_i + \mathfrak{d}\nu_i + d_H\rho_{ij} = \mathfrak{d}(\nu_i + \epsilon_i + d_H\omega_i)$ , i.e.  $d_H\rho_{ij} = d_H\mathfrak{d}\omega_i$ . Notice that if  $d_H\zeta_{ij} = 0$ , i.e. if  $\Xi$  is a symmetry of  $\mathfrak{d}\lambda_i$ , we have  $\mathcal{L}_\Xi d_H\gamma_{ij} = 0$ , then  $d_H\mathfrak{d}\beta_i = 0$ , which means  $\mathfrak{d}\beta_i = d_H\psi_{ij}$ ; thus finally we can write  $\mathfrak{d}(\nu_i + \epsilon_i) = d_H(\psi_{ij} - \mathfrak{d}\omega_i) = d_H(\psi_{ij} - \rho_{ij})$ .  $\square$

**Lemma 3** *Let  $\Xi$  be such that  $\mathcal{L}_\Xi \eta_{\lambda_i} = 0$ . If  $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$  and  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$ , then we have the conservation law  $d_H\mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$ , where  $\mathcal{L}_\Xi(\nu_i + \epsilon_i)$  is a local representative of a global conserved current.*

PROOF. In fact, since we are supposing  $\Xi$  being a *generalized symmetry*, we have

$$\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i).$$

On the other hand  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$  implies  $\mathfrak{d}d_H(\nu_i + \epsilon_i) = 0$ , so that the local problem defined by  $\mathcal{L}_\Xi(\nu_i + \epsilon_i)$  is variationally equivalent to the problem defined by the the global current  $\Xi_H \rfloor \mu_{\nu+\epsilon} + \Xi_V \rfloor p_{d_V \mu_{\nu+\epsilon}} \equiv \Xi_H \rfloor d_H(\nu_i + \epsilon_i) + \Xi_V \rfloor p_{d_V(d_H(\nu_i + \epsilon_i))}$ .

**Remark 4** Notice that the condition  $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$  means that the symmetry  $\Xi$  is not only a generalized symmetry, but it is required to be *also a symmetry of the local variational problem*  $\mathcal{L}_\Xi \lambda_i$ .  $\square$

Definitively, the variational approach to the study of invariance properties of equations not only provides conserved quantities associated with the invariance of field equations; when such equations are Euler-Lagrange equations of some (possibly local) Lagrangian we can make a more precise statement about the nature of such conserved quantities: *under the condition stressed in the above Remark*, they are determined by variations of strong Noether currents for the associated local system of Lagrangians. Even in the case of a local presentation of a variational problem, since  $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$ , such a variation of local strong Noether currents is equivalent to a global conserved current. This is an improvement to approaches to mechanics and field theory based on Lie's invariance theory of dynamical systems.

Thus consequently we state the following main result.

**Theorem 1** *The conserved current associated with a generalized symmetry, assumed to be also a symmetry of the variational derivative of the corresponding local inverse problem, is variationally equivalent to the variation of the strong Noether currents for the corresponding local system of Lagrangians. Since the variational Lie derivative of the local system of Lagrangians is a global object, such a variation is variationally equivalent to a global conserved current.*

## Acknowledgements

Research partially supported by the University of Torino through the Department of Mathematics research project *Metodi Geometrici in Fisica Matematica e Applicazioni*; E.W. is also supported by his Department of Mathematics research grant *Sequenze variazionali e Teoremi di Noether* 2010-2012.

## References

- [1] I.M. Anderson, T. Duchamp: On the existence of global variational principles, *Amer. Math. J.* **102** (1980) 781–868.
- [2] D.Bashkirov, G. Giachetta, L.Mangiarotti, G.Sardanashvily: Noether's second theorem in a general setting. Reducible gauge theories, *J .Phys.* **A38** (2005) 5329–5344.
- [3] E. Bessel-Hagen: Über die Erhaltungssätze der Elektrodynamik, *Math. Ann.* **84** (1921) 258–276.
- [4] A. Borowiec, M. Ferraris, M. Francaviglia, M. Palese: Conservation laws for non-global Lagrangians, *Univ. Iagel. Acta Math.* **41** (2003) 319–331.



- [5] P. Dedecker and W. M. Tulczyjew: Spectral sequences and the inverse problem of the calculus of variations, in *Lecture Notes in Mathematics* **836**, Springer-Verlag (1980), 498–503.
- [6] D. J. Eck: Gauge-natural bundles and generalized gauge theories, *Mem. Amer. Math. Soc.* **247** (1981) 1–48.
- [7] L. Fatibene, M. Ferraris, M. Francaviglia: Augmented variational principles and relative conservation laws in classical field theory, *Int. J. Geom. Methods Mod. Phys.* **2** (3) (2005) 373–392.
- [8] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth: Canonical connections in gauge-natural field theories, *Int. J. Geom. Methods Mod. Phys.* **5** (6) (2008) 973–988.
- [9] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth: Gauge-natural Noether currents and connection fields, *Int. J. Geom. Methods Mod. Phys.* **8**(1) (2011) 177–185.
- [10] M. Ferraris, M. Francaviglia, M. Raiteri: Conserved Quantities from the Equations of Motion (with applications to natural and gauge natural theories of gravitation), *Class.Quant.Grav.* **20** (2003) 4043–4066.
- [11] M. Ferraris, M. Palese, E. Winterroth: Local variational problems and conservation laws, *Diff. Geom. Appl.* **29** (2011) S80–S85.
- [12] M. Francaviglia, M. Palese, E. Winterroth: Second variational derivative of gauge-natural invariant Lagrangians and conservation laws, in *Differential geometry and its applications*, Matfyzpress, Prague (2005), 591–604.
- [13] M. Francaviglia, M. Palese, R. Vitolo: Symmetries in finite order variational sequences, *Czech. Math. J.* **52(127)** (1) (2002) 197–213.
- [14] D. Krupka: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. Appl.*; J. Janyška, D. Krupka eds., World Sci. (Singapore, 1990) 236–254.
- [15] Th.H.J. Lepage: Sur les champ geodesiques du Calcul de Variations, I, II, *Bull. Acad. Roy. Belg., Cl. Sci.* **22** (1936) 716–729, 1036–1046.
- [16] E. Noether: Invariante Variationsprobleme, *Nachr. Ges. Wiss. Gött., Math. Phys. Kl.* **II** (1918) 235–257.
- [17] M. Palese, E. Winterroth: Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles, *Arch. Math. (Brno)* **41** (3) (2005) 289–310.
- [18] M. Palese, E. Winterroth: Covariant gauge-natural conservation laws, *Rep. Math. Phys.* **54** (3) (2004) 349–364.

- [19] M. Palese, E. Winterroth: On the relation between the Jacobi morphism and the Hessian in gauge-natural field theories, *Teoret. Mat. Fiz.* **152** (2007), no. 2, 377–389; translation in *Theoret. and Math. Phys.* **152** (2007), no. 2, 1191–1200.
- [20] M. Palese, E. Winterroth: Variational Lie derivative and cohomology classes, *AIP Conf. Proc.* **1360** (2011) 106–112.
- [21] M. Palese, E. Winterroth, E. Garrone: Second variational derivative of local variational problems and conservation laws, *Arch. Math. (Brno)* **47**(5) (2011) 395–403.
- [22] G. Sardanashvily: Noether conservation laws issue from the gauge invariance of an Euler-Lagrange operator, but not a Lagrangian, arXiv:math-ph/0302012;
- [23] G. Sardanashvily: Gauge conservation laws in a general setting. Superpotential, *Int. J. Geom. Methods Mod. Phys.* **6** (6) (2009) 1047–1056.
- [24] F. Takens: A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979) 543–562.
- [25] W. M. Tulczyjew: The Lagrange Complex, *Bull. Soc. Math. France* **105** (1977) 419–431.
- [26] A. M. Vinogradov: On the algebro-geometric foundations of Lagrangian field theory, *Soviet Math. Dokl.* **18** (1977) 1200–1204.